Energy spectra for modified Rosen–Morse potential solved by the exact quantization rule

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Abstract
The exact quantization rule is efficient for calculating the energy spectra of all exactly solvable quantum systems. In this work, we calculate the energy spectrum of the Schrödinger equation with the modified Rosen–Morse potential by this quantization rule.

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1. Introduction

It is well known that the study of exactly solvable problems has attracted much attention since the early stage of quantum mechanics. The exact solutions of the Schrödinger equation for a hydrogen atom and a harmonic oscillator in three dimensions are the milestones in the development of quantum mechanics [1, 2]. Generally speaking, there are a few main methods of studying the solutions of quantum systems. The first is the so-called traditional method, that is, one solves the second-order differential equation to obtain the solutions of quantum systems [3]. This can be realized by transforming the Schrödinger equation to some ordinary differential equations, whose solutions are the special functions [2]. The second is the algebraic method. This can be realized by studying the Hamiltonian of quantum system. The latter is also related to the method of the supersymmetric quantum mechanics [4], further closely related with the factorization method [5]. The third is a method based on an exact quantization rule.

Recently, an exact quantization rule has been presented [6, 7] where there is a quantum correction term which is found to be the same for all bound states of an exactly solvable quantum system such that the energy levels of bound states can be calculated from the exact quantization rule. This method is effective for all exactly solvable quantum systems such as the

finite square well, the Morse potential and its generalization, the symmetric and asymmetric hyperbolic-type Rosen–Morse potentials, the first and second Pöschl–Teller potentials, the Hulthén potential, the harmonic oscillator and the hydrogen atom [6, 7]. Moreover, the rotating Morse potential with the Pekeris approximation [8], the modified harmonic oscillator, the Kratzer potential, the pseudoharmonic oscillator, the trigonometric Rosen–Morse potential and others were carried out by this exact quantization rule [9–11]. On the other hand, a good connection between this new quantization rule and the Langer correction has been established [12]. The purpose of this work is to apply this quantization rule to calculating the energy levels of the modified Rosen–Morse potential, which was not considered before.

This work is organized as follows. In section 2 the exact quantization rule is briefly reviewed. We apply this method to calculating the energy levels of the modified Rosen–Morse potential in section 3. Finally, some concluding remarks are given in section 4.

2. Exact quantization rule

We now give a brief review of this method [6, 7]. The one-dimensional Schrödinger equation is given by

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2M}{\hbar^2}[E - V(x)]\psi(x),\tag{1}$$

where the potential $V(x)$ is a piecewise continuous real function of the variable x. The Schrödinger equation is equivalent to the Riccati equation,

$$\frac{d}{dx}\phi(x) = -\frac{2M}{\hbar^2}[E - V(x)] - \phi(x)^2,\tag{2}$$

where $\phi(x) = \psi(x)^{-1}d\psi(x)/dx$ is the logarithmic derivative of wavefunction $\psi(x)$. As said by Yang in a talk on monopole: 'For the Sturm–Liouville problem, the fundamental trick is the definition of a phase angle which is monotonic with respect to the energy.' [13]. For the Schrödinger equation, the phase angle is the logarithmic derivative $\phi(x)$. From equation (2), the $\phi(x)$ decreases monotonically with respect to x between two turning points, where $E \geqslant V(x)$. Specifically, as x increases across a node of wavefunction $\psi(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$ and then decreases again. This exact quantization rule was shown [6, 7] for the one-dimensional Schrödinger equation as

$$\int_{x_A}^{x_B} k(x)\,dx = N\pi + Q, \qquad Q = \int_{x_A}^{x_B} \phi(x)\left[\frac{dk(x)}{dx}\right]\left[\frac{d\phi(x)}{dx}\right]^{-1} dx,\tag{3}$$

$$k(x) = \frac{\sqrt{2M[E - V(x)]}}{\hbar}, \qquad E \geqslant V(x),\tag{4}$$

where x_A and x_B are two turning points determined by $E = V(x)$. $N = n + 1$ is the number of nodes of $\phi(x)$ in the region $E \geqslant V(x)$ and it is larger by one than the number n of nodes of the wavefunction $\psi(x)$. The first term $N\pi$ is the contribution from the nodes of the logarithmic derivative of the wavefunction, and the second is called the quantum correction Q. It is found that, for all well-known exactly solvable quantum systems, this quantum correction Q is independent of the number of nodes of the wavefunction such that Q can be calculated from the ground state

$$Q = Q_0 = \int_{x_A}^{x_B} \phi_0(x)\left[\frac{dk_0(x)}{dx}\right]\left[\frac{d\phi_0(x)}{dx}\right]^{-1} dx,\tag{5}$$

where the subscription '0' denotes the ground state. However, the momentum $k(x)$ on the left-hand side of the exact quantization rule (3) is concerned with the energy levels E_n. That is to say, the energy levels of the exactly solvable system can be calculated simply from its ground state. This exact quantization rule was also generalized to the three-dimensional Schrödinger equation with a spherically symmetric potential [6, 7]. As what follows, we shall apply this method to calculating the energy levels of the modified Rosen–Morse potential, which was not studied before.

3. Modified Rosen–Morse potential

The modified Rosen–Morse potential [14] is given by

$$V(x) = -\frac{U_0 - U_1 \sinh(x/a)}{\cosh^2(x/a)}, \qquad -\infty < x < \infty. \tag{6}$$

$V(x)$ is the symmetrical Rosen–Morse potential if $U_1 = 0$. It should be noted that the q-deformed Rosen–Morse potential was considered by Grosche by using the path integral method [15]. Since $V(x)$ tends to zero as x goes to $\pm\infty$, the energy E_n of the bound state of the system, if exists, has to be negative. Let

$$y = \sinh(x/a), \qquad -\infty \leqslant y \leqslant \infty, \qquad \frac{dy}{dx} = \frac{\sqrt{1+y^2}}{a}, \qquad V(x) = -\frac{U_0 - U_1 y}{1 + y^2}. \tag{7}$$

The Riccati equation for the ground state is written in the new variable y as

$$\frac{\sqrt{1+y^2}}{a} \frac{d\phi_0(x)}{dy} = -\frac{2M}{\hbar^2}\left[E_0 + \frac{U_0 - U_1 y}{1 + y^2}\right] - \phi_0(x)^2. \tag{8}$$

Since the logarithmic derivative $\phi_0(x)$ for the ground state has one zero and no pole, it has to take the linear form in y. In addition, in order to meet the term with $\sqrt{1+y^2}$ in the Riccati equation (8), $\phi_0(x)$ can contain an additional factor $(1 + y^2)^{-1/2}$, which has no zero in the domain of definition of y,

$$\phi_0(x) = -\frac{Ay + B}{a\sqrt{1+y^2}}, \qquad A > 0. \tag{9}$$

Substituting equation (9) into equation (8), one has

$$-\frac{A - By}{a^2(1+y^2)} = -\frac{2M[E_0(1+y^2) + U_0 - U_1 y]}{\hbar^2(1+y^2)} - \frac{A^2 y^2 + 2ABy + B^2}{a^2(1+y^2)}. \tag{10}$$

Letting $G_0 = 2A + 1$, one has

$$E_0 = -\frac{\hbar^2(G_0 - 1)^2}{8Ma^2}, \qquad B = \frac{2Ma^2 U_1}{\hbar^2 G_0}$$

$$G_0^2 = 1 + \frac{8Ma^2 U_0}{\hbar^2} + 4B^2 = 1 + \frac{8Ma^2 U_0}{\hbar^2} + \left[\frac{4Ma^2 U_1}{\hbar^2 G_0}\right]^2. \tag{11}$$

Then,

$$G_0^2 = \frac{1}{2} + \frac{4Ma^2 U_0}{\hbar^2} + \left[\left(\frac{1}{2} + \frac{4Ma^2 U_0}{\hbar^2}\right)^2 + \left(\frac{4Ma^2 U_1}{\hbar^2}\right)^2\right]^{1/2},$$

$$B = \frac{U_1}{|U_1|}\left\{-\frac{1}{8} - \frac{Ma^2 U_0}{\hbar^2} + \left[\left(\frac{1}{8} + \frac{Ma^2 U_0}{\hbar^2}\right)^2 + \left(\frac{Ma^2 U_1}{\hbar^2}\right)^2\right]^{1/2}\right\}^{1/2}. \tag{12}$$

The turning points are solved from $E_n = V(x)$

$$
\begin{aligned}
y_A &= \sinh(x_A/a) = (-2E_n)^{-1}\left\{-U_1 - \sqrt{U_1^2 - 4E_n(U_0 + E_n)}\right\} \\
y_B &= \sinh(x_B/a) = (-2E_n)^{-1}\left\{-U_1 + \sqrt{U_1^2 - 4E_n(U_0 + E_n)}\right\} \\
y_A + y_B &= U_1/E_n, \qquad y_A y_B = 1 + U_0/E_n.
\end{aligned}
\tag{13}
$$

Between two turning points, the momentum $k(x)$ is expressed as

$$
k(x) = \frac{\sqrt{-2ME_n}}{\hbar\sqrt{1 + y^2}}\sqrt{(y_B - y)(y - y_A)}.
\tag{14}
$$

We are now in the position to derive the energy spectrum in terms of the exact quantization rule. During the calculation, the following integral formulae are useful:

$$
\int_{x_A}^{x_B} \frac{dx}{\sqrt{(x_B - x)(x - x_A)}} = \pi,
\tag{15}
$$

$$
\int_{x_A}^{x_B} \frac{x\,dx}{\sqrt{(x_B - x)(x - x_A)}} = \frac{\pi}{2}(x_A + x_B)
\tag{16}
$$

$$
\int_{x_A}^{x_B} \frac{x^2\,dx}{\sqrt{(x_B - x)(x - x_A)}} = \frac{\pi}{8}\left(3x_A^2 + 2x_A x_B + 3x_B^2\right)
\tag{17}
$$

$$
\int_{x_A}^{x_B} \frac{dx}{(a + bx)\sqrt{(x_B - x)(x - x_A)}} = \frac{\pi\lambda}{\sqrt{(a + bx_B)(a + bx_A)}},
$$
$$
a + bx \neq 0 \qquad \text{when} \quad x_A < x < x_B,
\tag{18}
$$

where $\lambda = \pm 1$ depends on the sign of $a + bx$.

The quantum correction Q_0 in the exact quantization rule can be calculated for the ground state

$$
\frac{\phi_0(x)}{d\phi_0(x)/dy} = -\frac{[(G_0 - 1)y + 2B](1 + y^2)}{2By - (G_0 - 1)},
$$
$$
\frac{dk(x)}{dy} = \frac{\sqrt{-2ME_0}}{2\hbar}\frac{(-2y + y_A + y_B)(1 + y^2) - 2y(y_B - y)(y - y_A)}{(1 + y^2)^{3/2}\sqrt{(y_B - y)(y - y_A)}}
$$
$$
= -\frac{\sqrt{-2ME_0}}{2\hbar}\frac{(y_A + y_B)(y^2 - 1) - 2(y_A y_B - 1)y}{(1 + y^2)^{3/2}\sqrt{(y_B - y)(y - y_A)}}
$$
$$
Q_0 = \frac{a\sqrt{-2ME_0}}{2\hbar}\int_{y_A}^{y_B}\frac{[(G_0 - 1)y + 2B]}{[2By - (G_0 - 1)](1 + y^2)}
$$
$$
\times \frac{[(y_A + y_B)(y^2 - 1) - 2(y_A y_B - 1)y]\,dy}{\sqrt{(y_B - y)(y - y_A)}}
$$
$$
= \frac{a\sqrt{-2ME_0}}{2\hbar}\int_{y_A}^{y_B}\frac{dy}{\sqrt{(y_B - y)(y - y_A)}}\left\{\frac{(G_0 - 1)(y_A + y_B)}{2B}\right.
$$
$$
+ \frac{[(G_0 - 1)^2 - 4B^2](y_A + y_B)/(2B) - 2(G_0 - 1)(y_A y_B - 1)}{2By - (G_0 - 1)}
$$
$$
\left. + \frac{2(y_A + y_B)y - 2(y_A y_B - 1)}{(y^2 + 1)}\right\}
$$
$$
= I_1 + I_2 + I_3.
\tag{19}
$$

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Note that for the ground state

$$y_A y_B - 1 = \frac{U_0}{E_0},$$

$$y_A + y_B = \frac{U_1}{E_0} = \frac{\hbar^2 G_0 B}{2Ma^2 E_0}, \tag{20}$$

$$\sqrt{(y_B + \mathrm{i})(y_A + \mathrm{i})} = \left[\frac{U_0}{E_0} + \mathrm{i}\frac{U_1}{E_0}\right]^{1/2},$$

and from equation (11)

$$\sqrt{[2By_A - (G_0 - 1)][2By_B - (G_0 - 1)]}$$

$$= \left[4B^2\left(1 + \frac{U_0}{E_0}\right) - \frac{2BU_1}{E_0}(G_0 - 1) + (G_0 - 1)^2\right]^{1/2}$$

$$= \frac{1}{\sqrt{-E_0}}\left\{-E_0[4B^2 + (G_0 - 1)^2] + 4B^2\left[\frac{\hbar^2 G_0(G_0 - 1)}{4Ma^2} - U_0\right]\right\}^{1/2}$$

$$= \frac{1}{\sqrt{-E_0}}\left\{\frac{\hbar^2(G_0 - 1)^2}{8Ma^2}\left[2G_0(G_0 - 1) - \frac{8Ma^2 U_0}{\hbar^2}\right]\right.$$

$$\left. + \left[G_0^2 - 1 - \frac{8Ma^2 U_0}{\hbar^2}\right]\left[\frac{\hbar^2 G_0(G_0 - 1)}{4Ma^2} - U_0\right]\right\}^{1/2}$$

$$= \frac{1}{\sqrt{-E_0}}\left\{\frac{\hbar^2 G_0^2(G_0 - 1)^2}{2Ma^2} - 4U_0 G_0(G_0 - 1) + \frac{8Ma^2 U_0^2}{\hbar^2}\right\}^{1/2}$$

$$= \frac{\hbar}{a\sqrt{-2ME_0}}\left|G_0(G_0 - 1) - \frac{4Ma^2 U_0}{\hbar^2}\right|. \tag{21}$$

Now, one is able to calculate the integrals I_1, I_2 and I_3:

$$I_1 = \frac{a\sqrt{-2ME_0}}{4\hbar}\int_{y_A}^{y_B}\mathrm{d}y\,\frac{(G_0 - 1)(y_A + y_B)}{B\sqrt{(y_B - y)(y - y_A)}}$$

$$= -\frac{\hbar G_0(G_0 - 1)}{4a\sqrt{-2ME_0}}\int_{y_A}^{y_B}\frac{\mathrm{d}y}{\sqrt{(y_B - y)(y - y_A)}}$$

$$= -\frac{\pi G_0}{2}. \tag{22}$$

$$[(G_0 - 1)^2 - 4B^2](y_A + y_B)/(2B) - 2(G_0 - 1)(y_A y_B - 1)$$

$$= \frac{1}{-E_0}\left\{\left[2(G_0 - 1) - \frac{8Ma^2 U_0}{\hbar^2}\right]\frac{\hbar^2 G_0}{4Ma^2} + 2(G_0 - 1)U_0\right\}$$

$$= \frac{1}{-E_0}\left\{\frac{\hbar^2 G_0(G_0 - 1)}{2Ma^2} - 2U_0\right\}, \tag{23}$$

$$I_2 = \frac{a\sqrt{-2ME_0}}{2\hbar}\int_{y_A}^{y_B}\mathrm{d}y \cdot \frac{[(G_0 - 1)^2 - 4B^2](y_A + y_B)/(2B) - 2(G_0 - 1)(y_A y_B - 1)}{[2By - (G_0 - 1)]\sqrt{(y_B - y)(y - y_A)}}$$

$$= \pm\pi\frac{a\sqrt{-2ME_0}}{2\hbar}\frac{\frac{1}{-E_0}\left\{\frac{\hbar^2 G_0(G_0 - 1)}{2Ma^2} - 2U_0\right\}}{\frac{\hbar}{a\sqrt{-2ME_0}}\left\{G_0(G_0 - 1) - \frac{4Ma^2 U_0}{\hbar^2}\right\}}$$

$$= \pm\frac{\pi}{2}. \tag{24}$$

We are going to determine the sign of I_2. When $U_1 = 0$, from equation (11) one has $G_0^2 = 1 + 8Ma^2U_0/\hbar^2$, and then

$$G_0(G_0 - 1) - \frac{4Ma^2U_0}{\hbar^2} = 1 + \frac{4Ma^2U_0}{\hbar^2} - \sqrt{1 + \frac{8Ma^2U_0}{\hbar^2}} > 0. \tag{25}$$

When $U_1 = 0$, the numerator of the integral of I_2 is positive, but its denominator is negative such that $I_2 < 0$. Since I_2 is a constant independent of U_1, one has

$$I_2 = -\frac{\pi}{2}. \tag{26}$$

The integral I_3 is calculated as

$$
\begin{aligned}
I_3 &= \frac{a\sqrt{-2ME_0}}{\hbar} \int_{y_A}^{y_B} \mathrm{d}y \, \frac{(y_A + y_B)y - (y_A y_B - 1)}{(y^2 + 1)\sqrt{(y_B - y)(y - y_A)}} \\
&= \frac{a\sqrt{-2ME_0}}{\hbar} \mathrm{Re} \int_{y_A}^{y_B} \mathrm{d}y \, \frac{(y_A + y_B) - \mathrm{i}(y_A y_B - 1)}{(y + \mathrm{i})\sqrt{(y_B - y)(y - y_A)}} \\
&= \pm \pi \frac{a\sqrt{-2ME_0}}{\hbar} \mathrm{Re}\{-\mathrm{i}\sqrt{(y_B + \mathrm{i})(y_A + \mathrm{i})}\} \\
&= \pm \pi \frac{a\sqrt{2M}}{\hbar} \mathrm{Re}\sqrt{U_0 + \mathrm{i}U_1},
\end{aligned}
\tag{27}
$$

where equation (20) is used. Letting $U_0 + \mathrm{i}U_1 = \rho \, \mathrm{e}^{\mathrm{i}\varphi}$, where $\rho = \sqrt{U_0^2 + U_1^2}$ and $\cos\varphi = U_0/\rho$, one has

$$\mathrm{Re}\sqrt{U_0 + \mathrm{i}U_1} = \sqrt{\rho}\cos(\varphi/2) = \sqrt{\rho}\sqrt{\frac{1 + \cos\varphi}{2}} = \sqrt{\frac{(U_0^2 + U_1^2)^{1/2} + U_0}{2}}. \tag{28}$$

There is a sign confusion in I_3 of equation (27) which comes from the square root. The sign can be determined by the continuous condition. In fact, as shown in equation (27), I_3 is positive when $U_1 = 0$ such that

$$I_3 = \frac{a\pi}{\hbar}\sqrt{M\left[(U_0^2 + U_1^2)^{1/2} + U_0\right]}. \tag{29}$$

Under the assumption $Q = Q_0$, one has from the exact quantization rule where $N = n + 1$ for the nth excited state,

$$\int_{x_A}^{x_B} k(x)\,\mathrm{d}x = -\frac{(G_0 - 2n - 1)\pi}{2} + \frac{a\pi}{\hbar}\sqrt{M\left[(U_0^2 + U_1^2)^{1/2} + U_0\right]}. \tag{30}$$

On the other hand, one has

$$
\begin{aligned}
\int_{x_A}^{x_B} k(x)\,\mathrm{d}x &= \frac{a\sqrt{-2ME_n}}{\hbar} \int_{y_A}^{y_B} \frac{\sqrt{(y_B - y)(y - y_A)}}{y^2 + 1}\,\mathrm{d}y \\
&= \frac{a\sqrt{-2ME_n}}{\hbar} \int_{y_A}^{y_B} \frac{-(y^2 + 1) + (y_A + y_B)y - (y_A y_B - 1)}{(y^2 + 1)\sqrt{(y_B - y)(y - y_A)}}\,\mathrm{d}y \\
&= -\pi\frac{a\sqrt{-2ME_n}}{\hbar} + I_4, \\
I_4 &= \frac{a\sqrt{-2ME_n}}{\hbar} \int_{y_A}^{y_B} \frac{(y_A + y_B)y - (y_A y_B - 1)}{(y^2 + 1)\sqrt{(y_B - y)(y - y_A)}}\,\mathrm{d}y \\
&= \frac{a\sqrt{-2ME_n}}{\hbar} \mathrm{Re} \int_{y_A}^{y_B} \frac{(y_A + y_B) - \mathrm{i}(y_A y_B - 1)}{(y + \mathrm{i})\sqrt{(y_B - y)(y - y_A)}}\,\mathrm{d}y
\end{aligned}
$$

$$= \pi \frac{a\sqrt{-2ME_n}}{\hbar} \mathrm{Re}\{-\mathrm{i}\sqrt{(y_B + \mathrm{i})(y_A + \mathrm{i})}\}$$

$$= \pi \frac{a\sqrt{2M}}{\hbar} \mathrm{Re}\sqrt{U_0 + \mathrm{i}U_1}$$

$$= \frac{a\pi}{\hbar}\sqrt{M\left[(U_0^2 + U_1^2)^{1/2} + U_0\right]}. \tag{31}$$

The sign of I_4 is determined by the same reason as the sign of I_3. In comparison of equations (30) and (31), one obtains

$$\frac{(G_0 - 2n - 1)}{2} = \frac{a\sqrt{-2ME_n}}{\hbar}. \tag{32}$$

Then, $G_0 > 2n + 1$ and

$$E_n = -\frac{\hbar^2 (G_0 - 2n - 1)^2}{8Ma^2}. \tag{33}$$

This result coincides with the previous calculation given in [14], which checks that $Q = Q_0$ is an invariant for the exactly solvable system.

4. Concluding remarks

We have calculated the energy levels of the Schrödinger equation with the modified Rosen–Morse potential by this exact quantization rule. The logarithmic derivative $\phi_0(x)$ of the wavefunction for the ground state is explicitly solved from the Riccati equation. It is shown that we need not use the exact eigenfunctions of this system to obtain the energy levels. In contrast, for an exactly solvable quantum system, its energy levels can be calculated only from the ground state. The calculation here and those in [6–12] show that this method is efficient for all exactly solvable quantum systems.

Acknowledgments

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